Exact quantization of a PT symmetric (reversible) Liénard type nonlinear oscillator

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Abstract. We carry out an exact quantization of a PT symmetric (reversible) Liénard type one dimensional nonlinear oscillator both semiclassically and quantum mechanically. The associated time independent classical Hamiltonian is of non-standard type and is invariant under a combined coordinate reflection and time reversal transformation. We use von Roos symmetric ordering procedure to write down the appropriate quantum Hamiltonian. While the quantum problem cannot be tackled in coordinate space, we show how the problem can be successfully solved in momentum space by solving the underlying Schrödinger equation therein. We obtain explicitly the eigenvalues and eigenfunctions (in momentum space) and deduce the remarkable result that the spectrum agrees exactly with that of the linear harmonic oscillator, which is also confirmed by a semiclassical modified Bohr-Sommerfeld quantization rule, while the eigenfunctions are completely different.

1. Introduction

In a previous paper [1], Chandrasekar and two of the present authors have presented a conservative description for the Liénard type one dimensional nonlinear oscillator, namely

$$\ddot{x} + kx\dot{x} + \frac{k^2}{9}x^3 + \omega^2 x = 0, \tag{1}$$

where overdot denotes differentiation with respect to t and k and ω^2 are real parameters. Expressing (1) as a sysytem of first order equations, $\dot{x} = y \equiv F_1(x,y)$, $\dot{y} = -kxy - \frac{k^2}{9}x^3 - \omega^2 x \equiv F_2(x,y)$, one can note that the divergence of the flow function $\vec{F} = F_1\vec{i} + F_2\vec{j}$ of (1) is non-zero ($\vec{\nabla}.\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} = -kx$). Also equation (1) is invariant under the PT or reversible transformation, $x \to -x$ and $t \to -t$ [2]. In spite of these, system (1) admits a time independent Hamiltonian of the form

$$H(x,p) = \frac{9\omega^4}{2k^2}\left[2 - \frac{2k}{3\omega^2}p - 2\left(1 - \frac{2k}{3\omega^2}p\right)^{\frac{1}{2}} + \frac{k^2x^2}{9\omega^2}\left(1 - \frac{2k}{3\omega^2}p\right)\right], \; -\infty < p \le \frac{3\omega^2}{2k}, \tag{2}$$

which is of non-standard type, that is the coordinates and potentials are mixed so that the Hamiltonian cannot be written as just the sum of the kinetic and potential energy terms alone, including velocity dependent terms. The corresponding Lagrangian L is given by

$$L = \frac{27\omega^6}{2k^2}\left(\frac{1}{k\dot{x} + \frac{k^2}{3}x^2 + 3\omega^2}\right) + \frac{3\omega^2}{2k}\dot{x} - \frac{9\omega^4}{2k^2}, \tag{3}$$

and the conjugate momentum is

$$p = \frac{\partial L}{\partial \dot{x}} = -\frac{27\omega^6}{2k(k\dot{x} + \frac{k^2}{3}x^2 + 3\omega^2)^2} + \frac{3\omega^2}{2k}. \tag{4}$$

We note here that the system (1) also admits an alternate Lagrangian/ Hamiltonian (see for example Ref. [1]). However we consider the Hamiltonian given in the form (2) only since as $k \to 0$ the Hamiltonian (2) reduces to the linear harmonic oscillator Hamiltonian, as the equation (1) does. We mention here that the parameters k and ω^2 can be rescaled with appropriate scaling in x and t. However, to describe the physical properties of this system in the classical, semi-classical and quantum levels, we retain the parameters k and ω^2 and do not scale them away. We also note here that the Hamiltonian (2) with the definition of p given by (4) is also invariant under combined action of coordinate reflection and time reversal (PT), $x \to -x$ and $t \to -t$.

The nonlinear oscillator (2) admits general periodic solution of the form

$$x(t) = \frac{A\sin(\omega t + \delta)}{1 - \frac{kA}{3\omega}\cos(\omega t + \delta)}, \qquad 0 \le A < \frac{3\omega}{k}, \tag{5}$$

where A, δ are arbitrary constants. Note that for $0 \le A < \frac{3\omega}{k}$ (while $-\infty < x < \infty$), the system (1) admits isochronous oscillations of frequency ω, which is the same as that of the linear harmonic oscillator. For $A \ge \frac{3\omega}{k}$, the solution becomes singular whenever the

phase $(\omega t + \delta)$ takes the value $\cos^{-1}\left(\frac{3\omega}{kA}\right) + 2n\pi,\ n$: any integer, even though $x(t)$ given by the function in (5) is periodic of period $T = \frac{2\pi}{\omega}$, while the corresponding momentum p is bounded and periodic (see equation (7) below). For more details on the classical dynamics of this system one may refer to Ref. [1].

Exactly solvable quantum mechanical problems, particularly the ones involving nonlinear potentials are rare, even in one dimension. The few examples include Pöschl-Teller, Morse, Scarf and isotonic oscillator potentials [3]. Also there exists a few velocity dependent potentials, for example Mathews-Lakshmanan oscillator and its generalizations [4, 5]. The quantization of (1) is a challenging problem since the obstacles in this task are many. For example, the quantization of the damped linear harmonic oscillator itself is a quite complicated procedure requiring a rigged Hilbert space [6] description whereas the system under consideration is a nonlinear one. In addition to this, the associated time independent Hamiltonian is a non-standard one [7]. To the authors' knowledge there exists no nonstandard Hamiltonian system which is quantum mechanically exactly solvable. Such a system cannot also be quantized using standard techniques of canonical quantization [8–10]. In this paper, we completely solve the quantum mechanical problem of the Liénard type oscillator (1), possessing the nonstandard Hamiltonian structure (2), by associating it with a position dependent mass Hamiltonian where now the variables x and p are interchanged. We then consider a general symmetric ordered form of the Hamiltonian proposed by von Roos [11] and solve the underlying Schrödinger equation in the momentum space. Since allowable choices of symmetric ordering lead to singular/unbounded solution, we transform the symmetric ordered Hamiltonian suitably in such a way that the associated Schrödinger equation possesses acceptable eigenfunctions. It is worth noting that the transformed Hamiltonian is now a non-symmetric ordered one as well as non-Hermitian. But it admits a real energy spectrum since the Hamiltonian and the corresponding eigenfunctions are invariant under the action of PT operation when $-\infty < p < \frac{3\omega^2}{2k}$. Our results reveal that the eigenvalues of (2) exactly match with that of the linear harmonic oscillator, though the eigenfunctions are of a more complicated nature in the momentum space. The explicit form of the eigenfunctions is also presented. We also obtain the energy level spectrum through a semiclassical modified Bohr-Sommerfeld quantization rule for the regular periodic solution (5) which agrees with the quantum mechanical results. Additionally, we point out the existence of a negative energy spectrum in the quantum case corresponding to the sector $p > \frac{3\omega^2}{2k}$.

We also note here the interesting fact that while the standard PT symmetric systems considered extensively in the recent literature [12–14] all correspond to PT invariant complex potentials involving complex valued dynamical variables, the present Hamiltonian system (2) is PT symmetric and real where the dynamical variables are also real. The motivation here is more of exact quantization of a nonlinear dynamical system. Consequently the analysis of the corresponding quantum system in the momentum space discussed below will also be different in spirit from the modified normalization scheme of complexified PT-symmetric schemes [15].

2. Semiclassical quantization

To start with let us consider the semiclassical aspects. To quantize the system semiclassically, we use the modified Bohr-Sommerfeld quantization rule [16], namely

$$\oint p\, dx = (n + \frac{1}{2})h, \tag{6}$$

where h is the Plank's constant and n is any nonnegative integer and the integration is carried out over a closed orbit in the (x, p) space.

Firstly, we determine the energy of the system $E = H$ using the general solution (5) in (2). Plugging the expression (5) in (4), we obtain

$$p \;= A\omega\cos(\omega t + \delta)\left(1 - \frac{kA}{6\omega}\cos(\omega t + \delta)\right). \tag{7}$$

Note that for regular (non-singular bounded) periodic oscillations in $x(t)$ given by equation (5), the amplitude is restricted to the range $0 \le A < \frac{3\omega}{k}$, while the range of p is restricted to $-\frac{9\omega^2}{2k} < p < \frac{3\omega^2}{2k}$. As we mentioned earlier (below (5)) when $A \ge \frac{3\omega}{k}$, one has singular periodic solution and there is no lower bound on p. Substituting now the expressions (5) and (7) in (2), we find that the energy of the system turns out to be

$$E \;= \frac{1}{2}A^2\omega^2. \tag{8}$$

Now, to evaluate the integral (6), we use (7) for p, and express dx from (5) in the form

$$dx \;= \frac{\left(A\cos\phi - \frac{kA^2}{3\omega^2}\right)}{\left(1 - \frac{kA}{3\omega^2}\cos\phi\right)^2}d\phi, \qquad \phi = \omega t + \delta, \tag{9}$$

so as to obtain the quantization condition for the regular periodic orbits,

$$\omega A^2 \int_0^{2\pi}\left[\frac{\left(\cos\phi - \frac{kA}{6\omega}\cos^2\phi\right)\left(\cos\phi - \frac{kA}{3\omega}\right)}{\left(1 - \frac{kA}{3\omega}\cos\phi\right)^2}\right]d\phi = (n + \frac{1}{2})h. \tag{10}$$

Evaluating the above integral, we arrive at

$$A^2\omega = 2\left(n + \frac{1}{2}\right)\hbar, \qquad 0 \le A < \frac{3\omega}{k}. \tag{11}$$

Finally, from equations (8) and (11), we obtain the allowed energy eigenvalues with an appropriate upper bound N on n corresponding to the regular periodic orbits of (2) as

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega, \qquad n = 0, 1, 2, ...N \tag{12}$$

which agrees with that of the linear harmonic oscillator for this part of the spectrum. The semiclassical approach motivates us to prove that the energy of the nonlinear oscillator (2) can be quantized exactly as given in equation (12). In the following, we proceed to solve the time indepedent Schrödinger equation associated with the system (2) analytically, not in the coordinate space but in the momentum space.

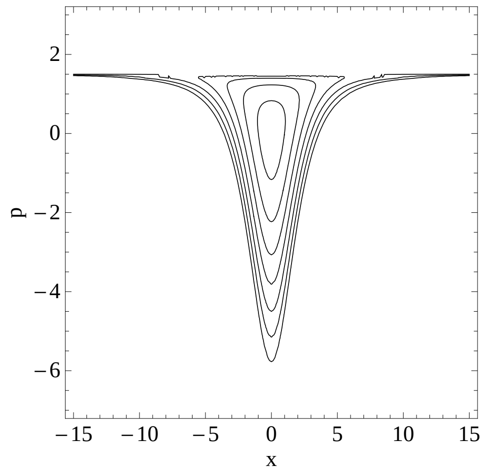

Figure 1. The phase trajectories of the Hamiltonian system (13) with $\omega = k = 1$ for various values of $E = H$.

3. Quantum exact solvability

Next we observe that the classical Hamiltonian $H(x, p)$ given in (2) is of the non-standard type, that is,

$$H(x,p) = \frac{1}{2} f(p) x^2 + U(p), \tag{13}$$

where

$$f(p) = \omega^2 \left(1 - \frac{2k}{3\omega^2} p\right), \qquad U(p) = \frac{9\omega^4}{2k^2} \left(\sqrt{1 - \frac{2k}{3\omega^2} p} - 1\right)^2. \tag{14}$$

The $(x - p)$ phase space structure is shown schematically in figure 1. Note the deformed nature of the bounded periodic orbits around the origin and $-\frac{9\omega^2}{2k} < p < \frac{3\omega^2}{2k}$. The remaining trajectories have only an upper bound at $p = \frac{3\omega^2}{2k}$.

Note that in the limit $k \to 0$, $f(p) \to \omega^2, U(p) \to \frac{p^2}{2}$, so that

$$H = \frac{p^2}{2} + \frac{\omega^2 x^2}{2} \tag{15}$$

as it should be. Now the first term in the Liénard oscillator Hamiltonian (13) contains both the position and momentum variables while the second term turns out to be a function of momentum alone. To quantize the Hamiltonian of this nature, one has to adopt a suitable ordering procedure. Since x and p are non-commuting variables in the quantum case, one may consider different ways of ordering between x and $f(p)$ in order to quantize this Hamiltonian. After performing a detailed analysis we find that the nonstandard classical Hamiltonian given in (2) can also be equivalently considered in the form

$$H(x,p) = \frac{x^2}{2\, m(p)} + U(p), \qquad -\infty < p \leq \frac{3\omega^2}{2k}, \tag{16}$$

where

$$m(p) = \frac{1}{\omega^2\ \left(1 - \frac{2k}{3\omega^2}p\right)} \quad \text{and} \quad U(p) = \frac{9\omega^4}{2k^2}\left(\sqrt{1 - \frac{2k}{3\omega^2}p} - 1\right)^2. \tag{17}$$

Interestingly, this form is similar to a position dependent mass Hamiltonian, $H = \frac{p^2}{2\,m(x)} + V(x)$, discussed extensively recently [11,17–19] but with an important difference that the variables x and p are now interchanged. Once this fact is recognized one can consider a general symmetric ordered form of the quantum Hamiltonian proposed by von Roos in order to quantize the position (but now actually momentum) dependent mass Schrödinger equation [11] as

$$H(\hat{x}, \hat{p}) = \frac{1}{4}\left[m^{\alpha}(\hat{p})\hat{x}m^{\beta}(\hat{p})\hat{x}m^{\gamma}(\hat{p}) + m^{\gamma}(\hat{p})\hat{x}m^{\beta}(\hat{p})\hat{x}m^{\alpha}(\hat{p})\right] + U(\hat{p}), \tag{18}$$

where the parameters α, β and γ which remain to be fixed have to satisfy the condition $\alpha+\beta+\gamma = -1$. Obviously now in (18) $\hat{x}$ and $\hat{p}$ are linear Hermitian operators satisfying the commutation rule

$$[\hat{x}, \hat{p}] = i\hbar. \tag{19}$$

We note here that since the variables $\hat{x}$ and $\hat{p}$ are interchanged we solve the Schrödinger equation corresponding to the Hamiltonian (18) in momentum space with $\hat{x} = i\hbar\dfrac{\partial}{\partial p}$ and obtain the time independent one dimensional Schrödinger equation in the form

$$H(\hat{x}, \hat{p})\psi(x, p) = E\psi(x, p), \tag{20}$$

or

$$\frac{-\hbar^2}{2m}\left[\psi'' - \frac{m'}{m}\psi' + \left(\frac{1+\beta}{2}\right)\left(2\frac{m'^2}{m^2} - \frac{m''}{m}\right)\psi + \frac{\alpha(\alpha+\beta+1)m'^2}{m^2}\psi\right] + U(p)\psi = E\psi, \tag{21}$$

where prime stands for differentiation with respect to p. Since we are looking for bound states (even when $U(p)$ becomes complex for $p > \frac{3\omega^2}{2k}$), we can choose

$$\psi = 0, \quad \text{for} \quad p \geq \frac{3\omega^2}{2k} \tag{22}$$

and concentrate on the region $-\infty < p \leq \frac{3\omega^2}{2k}$ alone in this section. We will also impose the boundary conditions

$$\psi(-\infty) = \psi\left(\frac{3\omega^2}{2k}\right) = 0 \tag{23}$$

on the eigenfunctions for continuity and boundedness.

Substituting now the expression for $m(p)$ from (17) and its derivatives in equation (21) and simplifying the resultant expression, we arrive at

$$\psi'' - \frac{2k}{3\omega^2}\frac{1}{\left(1 - \frac{2k}{3\omega^2}p\right)}\psi' + \frac{4k^2\alpha(\alpha+\beta+1)}{9\omega^4\ \left(1 - \frac{2k}{3\omega^2}p\right)^2}\psi - \frac{2Ek^2 - 9\omega^4\left(\sqrt{1 - \frac{2k}{3\omega^2}p} - 1\right)^2}{\hbar^2\omega^2k^2\left(1 - \frac{2k}{3\omega^2}p\right)}\psi$$
$$= 0, \qquad -\infty < p \leq \frac{3\omega^2}{2k}. \tag{24}$$

Equation (24) can be further simplified by introducing a transformation $y^2 = \left(1 - \dfrac{2k}{3\omega^2}p\right)$ or $y = \sqrt{1 - \dfrac{2k}{3\omega^2}p}$. Since $-\infty < p \le \frac{3\omega^2}{2k}$, we have $0 \le y < \infty$. The resulting simplification of (24) yields

$$\frac{d^2\psi}{dy^2} + \frac{1}{y}\frac{d\psi}{dy} + \left(\tilde{E} + \frac{4\alpha(\alpha+\beta+1)}{y^2} - a(y-1)^2\right)\psi = 0, \quad 0 \le y < \infty, \tag{25}$$

where we have defined $\dfrac{18\omega^2}{\hbar^2\, k^2}E = \tilde{E}$, and $\dfrac{3^4\omega^6}{\hbar^2\, k^4} = a$.

In order to solve (25) subject to the boundary conditions corresponding to (23), namely $\psi(-\infty) = \psi(0) = 0$, we first note the admissible asymptotic behaviour, $\psi(y) \to e^{-\frac{\sqrt{a}}{2}(y^2-2y)}$ as $y \to \infty$. So we introduce another transformation, namely

$$\psi(y) = e^{-\frac{\sqrt{a}}{2}y^2 + \sqrt{a}y}\, \phi(y) \tag{26}$$

in (25) so that it can be rewritten as

$$\frac{d^2\phi}{dy^2} + \left(2\sqrt{a} - 2\sqrt{a}y + \frac{1}{y}\right)\frac{d\phi}{dy} + \left(\frac{4\alpha(\alpha+\beta+1)}{y^2} + \frac{\sqrt{a}}{y} + \tilde{E} - 2\sqrt{a}\right)\phi = 0. \tag{27}$$

Equation (27) can now be transformed to the Hermite differential equation under the change of variables

$$\phi(y) = y^{-1/2}\chi(z), \qquad z = a^{1/4}(y-1), \tag{28}$$

with the condition $4\alpha(\alpha+\beta+1) = -\frac{1}{4}$. The transformed equation turns out to be of the form

$$\frac{d^2\chi}{dz^2} - 2z\frac{d\chi}{dz} + \left(\frac{\tilde{E} - \sqrt{a}}{\sqrt{a}}\right)\chi = 0. \tag{29}$$

With the restriction of the constant $\dfrac{\tilde{E} - \sqrt{a}}{\sqrt{a}} = 2n$, $n = 0, 1, 2, ...$, equation (29) becomes the standard differential equation for the Hermite polynomials,

$$\frac{d^2\chi}{dz^2} - 2z\frac{d\chi}{dz} + 2n\chi = 0, \qquad n = 0, 1, 2, ... \tag{30}$$

where $\chi = H_n(z)$ which are nothing but the Hermite polynomials [20]. Then the eigenfunctions and eigenvalues can be readily written down as

$$\psi_n(p) = N_n \frac{\exp\left(-\frac{9\omega^3}{2\,\hbar\, k^2}\left(1 - \frac{2k}{3\omega^2}p - 2\sqrt{1 - \frac{2k}{3\omega^2}p}\right)\right)}{\left(1 - \frac{2k}{3\omega^2}p\right)^{1/4}} \times H_n\left[\frac{3\omega^{3/2}}{\sqrt{\hbar}\, k}\left(\sqrt{1 - \frac{2k}{3\omega^2}p} - 1\right)\right], \; -\infty < p \le \frac{3\omega^2}{2k}, \tag{31}$$

$$= 0, \qquad \frac{3\omega^2}{2k} \le p < \infty, \tag{32}$$

and

$$E_n = (n + \frac{1}{2})\hbar\omega, \qquad n = 0, 1, 2, ... \tag{33}$$

Here N_n are constants.

Note that the above eigenfunction is singular at the boundary $p = \frac{3\omega^2}{2k}$ due to the denominator term in the right hand side of the equation (31) and so the eigenfunction becomes unbounded. To avoid this singularity in the eigenfunction, we modify the starting Hamiltonian, $H(\hat{x}, \hat{p})$, suitably and solve the associated Schrödinger equation. For this purpose we can rewrite the time independent Schrödinger equation $H\psi = E\psi$ as

$$H(m^{-d}\Phi) = E(m^{-d}\Phi), \tag{34}$$

so that we have

$$\tilde{H}\ \Phi = (m^d H m^{-d})\ \Phi = E\Phi \quad \text{and} \quad \Phi = m^d\ \psi, \quad m(p) = \frac{1}{\omega^2\left(1 - \frac{2k}{3\omega^2}p\right)}. \tag{35}$$

With the choice $d < -\frac{1}{4}$, one can have bounded, continuous and single valued wavefunction for the Hamiltonian $\tilde{H}$.

As a simple choice we consider a specific set of the values of the ordering parameters $\alpha = \gamma = -\frac{1}{4}$ and $\beta = -\frac{1}{2}$ that satisfy the conditions $\alpha + \beta + \gamma = -1$ and $4\alpha(\alpha + \beta + 1) = -\frac{1}{4}$ so that $d = -\frac{1}{2}$. Hence the Hamiltonian $H(\hat{x}, \hat{p})$ in (18) becomes

$$H(\hat{x}, \hat{p}) = \frac{1}{2}\left[m^{-1/4}(\hat{p})\hat{x}m^{-1/2}(\hat{p})\hat{x}m^{-1/4}(\hat{p})\right] + U(\hat{p}), \tag{36}$$

which admits the solution as given in (31). Now we transform the Hamiltonian, given in (36), to the form

$$\tilde{H} = \frac{1}{\sqrt{m}}H\sqrt{m} = \frac{1}{2}\left[m^{-3/4}(\hat{p})\hat{x}m^{-1/2}(\hat{p})\hat{x}m^{1/4}(\hat{p})\right] + U(\hat{p}), \tag{37}$$

$$= -\frac{\hbar^2}{2}\omega^2\left(1 - \frac{2k}{3\omega^2}p\right)\left[\frac{d^2}{dp^2} + \frac{k^2}{12\omega^4}\frac{1}{\left(1 - \frac{2k}{3\omega^2}p\right)^2}\right] + \frac{9\omega^4}{2k^2}\left(\sqrt{1 - \frac{2k}{3\omega^2}p} - 1\right)^2. \tag{38}$$

This Hamiltonian (38) is invariant under PT symmetry [2] though it is nonsymmetric and non-Hermitian. The Schrödinger equation corresponding to the Hamiltonian, $\tilde{H}(x, p)$ is

$$-\frac{\hbar^2\omega^2}{2}\left(1 - \frac{2k}{3\omega^2}p\right)\Phi'' - \frac{\hbar^2 k^2}{24\omega^2\left(1 - \frac{2k}{3\omega^2}p\right)}\Phi + \frac{9\omega^4}{2k^2}\left(\sqrt{1 - \frac{2k}{3\omega^2}p} - 1\right)^2\Phi = E\Phi, \quad \left(' = \frac{d}{dp}\right). \tag{39}$$

Equation (39) is now solved again by following the above procedure. The resulting bound state solution turns out to be

$$\Phi_n(p) = \begin{cases} \tilde{N}_n\left(1 - \frac{2k}{3\omega^2}p\right)^{1/4}\exp\left(-\frac{9\omega^3}{2\,\hbar\,k^2}\left(1 - \frac{2k}{3\omega^2}p - 2\sqrt{1 - \frac{2k}{3\omega^2}p}\right)\right) \\ \qquad \times H_n\left[\frac{3\omega^{3/2}}{\sqrt{\hbar}\,k}\left(\sqrt{1 - \frac{2k}{3\omega^2}p} - 1\right)\right], -\infty < p \leq \frac{3\omega^2}{2k}, \\ 0, \quad p \geq \frac{3\omega^2}{2k}, \end{cases} \tag{40}$$

and the corresponding energy eigenvalues continue to be

$$E_n = (n + \frac{1}{2})\ \hbar\,\omega, \qquad n = 0, 1, 2, ... \tag{41}$$

One can observe that the solution (40) is continuous, single valued and bounded in the entire region $-\infty < p < \infty$ and satisfy the boundary conditions $\Phi(\infty) = \Phi(3k/2\omega^2) = \Phi(-\infty) = 0$. Since the eigenfunctions $\Phi_n(p)$ are bounded and continuous in the region $-\infty < p \leq \frac{3\omega^2}{2k}$ and are zero outside this region, they are also normalizable. The normalization constants $\tilde{N}_n$ can be found using the integration

$$\int_{-\infty}^{\frac{3\omega^2}{2k}} \Phi_n^*(p)\ \Phi_n(p) dp = 1. \tag{42}$$

This can be evaluated as

$$\int_{-\infty}^{0} \Phi_n^*(p)\ \Phi_n(p) dp + \int_{0}^{\frac{3\omega^2}{2k}} \Phi_n^*(p)\ \Phi_n(p) dp = 1. \tag{43}$$

On evaluating (43) becomes

$$\tilde{N}_n^2 \sqrt{\hbar\omega}\ e^{\left(\frac{9\omega^3}{k^2\hbar}\right)} \ \left(2^{n-1}\sqrt{\pi} n! \left(1 + \frac{9\omega^3}{k^2\hbar}\right) + g(a)\right) = 1, \tag{44}$$

where

$$g(a) = \int_{0}^{\frac{3\omega^2}{2k}} \Phi_n^*(p)\ \Phi_n(p) dp. \tag{45}$$

Hence the normalization constant is

$$\tilde{N}_n = \left(\frac{e^{-\left(\frac{9\omega^3}{k^2\hbar}\right)}}{\sqrt{\hbar\omega}(2^{n-1}\sqrt{\pi} n! \left(1 + \frac{9\omega^3}{k^2\hbar}\right) + g(a))} \right)^{1/2}. \tag{46}$$

We further note that in the limit $k \to 0$, equation (40) reduces to

$$\Phi_n(p) = \left(\frac{-1}{2^n \sqrt{\pi}\ n!}\right)^{1/2} \exp\left(-\frac{1}{2\ \hbar\ \omega} p^2\right) H_n \left[\frac{1}{\sqrt{\hbar\omega}} p\right], \quad -\infty < p < \infty, \tag{47}$$

which matches with the bound state solution of the harmonic oscillator in accordance with its classical counter part (vide (15)).

Finally one can also note that one can choose many number of possible non-symmetric Hamiltonian $\tilde{H}$ in (37) for suitable choice of the set of parameters α, β, γ and d all of which lead to the same eigenvalue spectrum but different sets of eigenfunctions.

4. The $p > \frac{3\omega^2}{2k}$ sector: broken symmetry

In addition to the above solutions, we can also identify a different set of solutions which is nonzero only in the regime $p > \frac{3\omega^2}{2k}$ with a different set of boundary conditions than (23). To realize this, we consider solutions with $\Phi = 0$, $-\infty < p \leq \frac{3\omega^2}{2k}$, and look for acceptable solutions in the region $\frac{3\omega^2}{2k} < p \leq \infty$, either bound states with the boundary condition $\Phi(\frac{3\omega^2}{2k}) = 0 = \Phi(\infty)$ or continuum states or both. We also note that in the case of the classical nonlinear oscillator (1) with the Hamiltonian (2), there exists no real solution for $p > \frac{3\omega^2}{2k}$ due to the form of the conjugate momentum (4).

For the above purpose, we consider the Schrödinger equation (39) corresponding to the Hamiltonian (38), under the transformation $\tilde{y} = \sqrt{\frac{2k}{3\omega^2}p - 1}$, as

$$\frac{d^2\Phi}{d\tilde{y}^2} - \frac{1}{\tilde{y}}\frac{d\Phi}{d\tilde{y}} + \left(-\tilde{E} + \frac{3}{4\tilde{y}^2} + a(i\tilde{y} - 1)^2\right)\Phi = 0, \quad 0 \leq \tilde{y} < \infty, \tag{48}$$

where again we have defined $\dfrac{18\omega^2}{\hbar^2\ k^2}E = \tilde{E}$, and $\dfrac{3^4\omega^6}{\hbar^2\ k^4} = a$. Under the transformation $\Phi(\tilde{y}) = \sqrt{\tilde{y}}e^{-\frac{\sqrt{a}}{2}\tilde{y}^2 - i\sqrt{a}\tilde{y}}\chi(z)$ with $z = a^{1/4}(\tilde{y} + i)$, we get

$$\frac{d^2\chi(z)}{dz^2} - 2z\frac{d\chi(z)}{dz} - \frac{\tilde{E} + \sqrt{a}}{\sqrt{a}}\chi(z) = 0, \tag{49}$$

so that the bounded solution (as $|z| \to \infty$ *or* $\tilde{y} \to \infty$) can be now expressed (compared to (29)) as

$$\chi(z) = H_n(z), \quad \tilde{E} = \frac{18\omega^2}{\hbar^2 k^2}E = -(2n+1)\sqrt{a}, \quad n = 0, 1, 2, ... \tag{50}$$

Thus the second set of solution to the Schrödinger equation (39) can be written as

$$\Phi_n(p) = \begin{cases} \tilde{\mathcal{N}}_n \left(\frac{2k}{3\omega^2}p - 1\right)^{1/4} \exp\left(-\frac{9\omega^3}{2\,\hbar\,k^2}\left(\frac{2k}{3\omega^2}p - 1 + i\,2\sqrt{\frac{2k}{3\omega^2}p - 1}\right)\right) \\ \qquad\qquad \times H_n\left[\frac{3\omega^{3/2}}{\sqrt{\hbar}\,k}\left(\sqrt{\frac{2k}{3\omega^2}p - 1} + i\right)\right], \quad p \geq \frac{3\omega^2}{2k}, \\ 0, \quad -\infty < p \leq \frac{3\omega^2}{2k}, \end{cases} \tag{51}$$

with energy eigenvalues without a lower bound

$$E_n = -(n + \frac{1}{2})\hbar\omega, \quad n = 0, 1, 2, 3, ... \tag{52}$$

In (51) $\tilde{\mathcal{N}}_n$ is the normalization constant. Note that the eigenfunctions (51) are no longer PT symmetric, even though the Hamiltonian (38) is PT symmetric, leading to a negative energy spectrum that is unbounded below. Such a broken symmetry is obviously a consequence of imposition of a different set of boundary conditions for the sector $p > \frac{3\omega^2}{2k}$ than (23) which is reminiscent of the situation in the case of the linear harmonic oscillator [12, 21] and the $H = \hat{p}^2 - \hat{x}^4$ oscillator [12].

5. Conclusion

We have shown that the non-Hermitian Hamiltonian $\tilde{H}(\hat{x}, \hat{p})$ given by (37) admits the bound state solutions, $\Phi_n(p)$ and real energy eigen spectrum, E_n. The energy eigenvalues E_n also match with the energy values obtained through a semiclassical approach corresponding to regular periodic orbits. It is interesting to observe that the quantum system (36) possesses the energy eigenvalues E_n which are same as that of the linear harmonic oscillator, though the eigenfunctions are quite different from that of the linear harmonic oscillator. Our analysis shows that the underlying Liénard type PT-invariant reversible nonlinear oscillator is exactly quantizable and leads to interesting class of eigenfunctions and energy spectrum. It is also possible to generalize the above results to more general class of Liénard type nonlinear oscillators [22] and coupled nonlinear oscillators [23], which will be taken up elsewhere.

6. Acknowledgments

VC wishes to thank the Council of Scientific and Industrial Research, Government of India, for providing a Senior Research Fellowship. The work forms a part of a research project of MS, and an IRHPA project and a Ramanna Fellowship project of ML, sponsored by the Department of Science and Technology (DST), Government of India. ML also acknowledges the financial support under a DAE Raja Ramanna Fellowship.

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